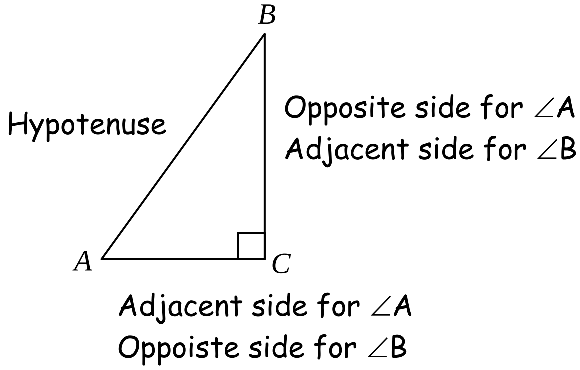
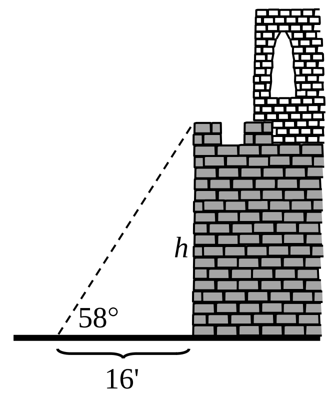
Geometry Notes Trigonometry – 1: Intro to Trig

First, some vocabulary:

Referring to an acute angle in a right triangle, we use the terminology at right.



Ex: In the year 1137, King Gudenuf plans to attack the castle of the evil Lord Fetid. He asks his royal mathematician, Trig, to figure out how high the walls of the castle are. Trig measured out a distance of 16 feet from the wall and then carefully measured the angle of elevation to be 58°. How can he find the height of the wall?

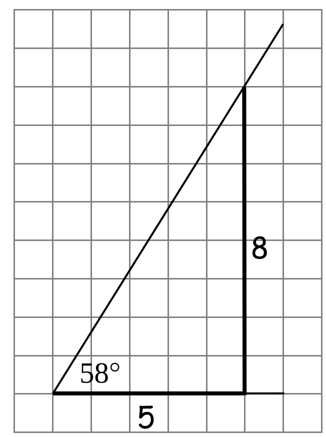


1. Find a piece of parchment. (Preferably graph parchment but any old parchment will do. Heck, he can even use an area of smooth ground.)

2. Carefully draw a 58° angle.

3. Measure *any convenient length* for the adjacent leg. Then draw the opposite leg. How does this triangle compare to the original?

   Trig chose 5 (you could choose a different length)



4. Measure the opposite leg.

   Almost exactly 8

5. Set up a proportion and solve.

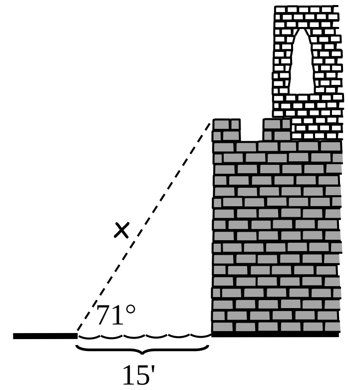
Note: The two triangles are similar by AA.

$$\frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}} \rightarrow \frac{h}{16} = \frac{8}{5}$$

5h = 128

h = 24.3 feet

Ex: King Gudenuf moved on to attack the castle of the vile Baron Malodorous. Malodorous has a 15 foot wide moat around his castle. Gudenuf asks Trig how long a ladder must be to reach from the far side of the moat to the top of the castle wall. Trig measured the angle of elevation to be 71°. What length of ladder is needed?



2. Carefully draw a 71° angle.

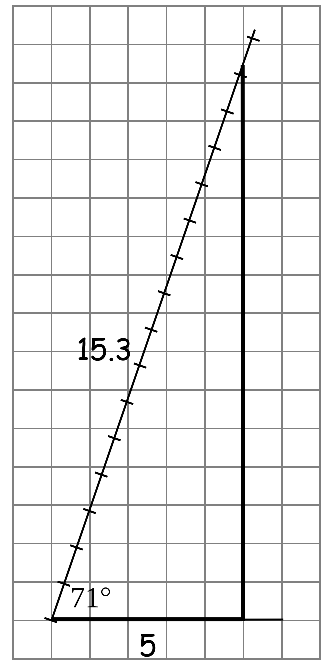
3. Measure *any convenient length* for the adjacent leg. Then draw the opposite leg.

   Trig chose 5 again (you could choose a different length)

4. Measure the hypotenuse.

   Trig got about 15.3

5. Set up a proportion and solve.



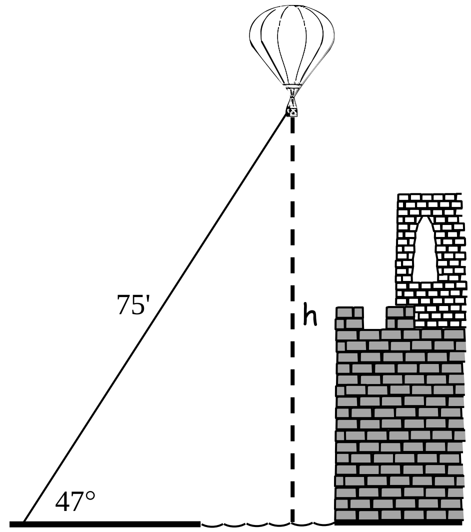
Note: Again, the two triangles are similar by AA.

$$\frac{\text{adj}}{\text{hyp}} = \frac{\text{adj}}{\text{hyp}} \rightarrow \frac{15}{x} = \frac{5}{15.3}$$

$$5x = 229.5$$

$$x = 45.9 \text{ feet}$$

Ex: King Gudenuf next planned to attack the castle of the foul Count Flatulent. Before attacking, he wanted information about how well Flatulent was prepared to defend his castle. With an idea about six and a half centuries before its time, Trig suggested sending a spy up in a hot air balloon. The balloon was tethered at the end of a 75 foot length of rope which made an angle of 47 with the ground. The King wanted to know how high was the balloon.



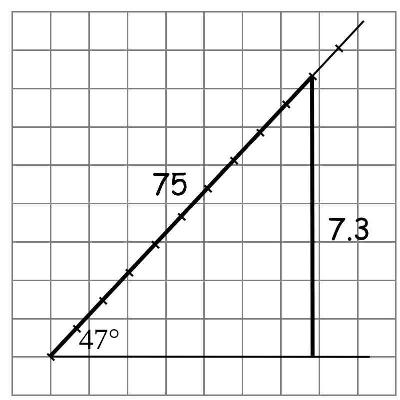
2. Carefully draw a 47° angle.

3. Measure *any convenient length* for the hypotenuse. Then draw the opposite leg.

   Trig chose 10 (you could choose a different length)

4. Measure the opposite leg.

   Trig got about 7.3



5. Set up a proportion and solve.

Note: Yet again, the two triangles are similar by AA.

$$\frac{\text{opp}}{\text{hyp}} = \frac{\text{opp}}{\text{hyp}} \rightarrow \frac{h}{75} = \frac{7.3}{10}$$

$10h = 547.5$

$x = 54.75$ feet

By now, Trig was really tired of drawing and measuring similar triangles so he could find the ratios he needed. He thought how great it would be if someone made a table of $\frac{\text{opp}}{\text{hyp}}$, $\frac{\text{adj}}{\text{hyp}}$ and $\frac{\text{opp}}{\text{adj}}$ for all the angles from 0 to 90°. He went to the King's library and found that some Greek guy named Hipparchus had the same idea over 1000 years before. Trig copied the table and took it home to practice some problems on.

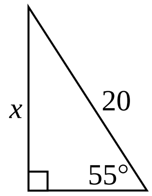
Table of Trigonometric Values

Angle	Opp/Hyp	Adj/Hyp	Opp/Adj	Angle	Opp/Hyp	Adj/Hyp	Opp/Adj
0	0.0000	1.0000	0.0000	46	0.7193	0.6947	1.0355
1	0.0175	0.9998	0.0175	47	0.7314	0.6820	1.0724
2	0.0349	0.9994	0.0349	48	0.7431	0.6691	1.1106
3	0.0523	0.9986	0.0524	49	0.7547	0.6561	1.1504
4	0.0698	0.9976	0.0699	50	0.7660	0.6428	1.1918
5	0.0872	0.9962	0.0875	51	0.7771	0.6293	1.2349
6	0.1045	0.9945	0.1051	52	0.7880	0.6157	1.2799
7	0.1219	0.9925	0.1228	53	0.7986	0.6018	1.3270
8	0.1392	0.9903	0.1405	54	0.8090	0.5878	1.3764
9	0.1564	0.9877	0.1584	55	0.8192	0.5736	1.4281
10	0.1736	0.9848	0.1763	56	0.8290	0.5592	1.4826
11	0.1908	0.9816	0.1944	57	0.8387	0.5446	1.5399
12	0.2079	0.9781	0.2126	58	0.8408	0.5299	1.6003
13	0.2250	0.9744	0.2309	59	0.8572	0.5150	1.6643
14	0.2419	0.9703	0.2493	60	0.8660	0.5000	1.7321
15	0.2588	0.9659	0.2679	61	0.8746	0.4848	1.8004
16	0.2756	0.9613	0.2867	62	0.8829	0.4695	1.8807
17	0.2924	0.9563	0.3057	63	0.8910	0.4540	1.9626
18	0.3090	0.9511	0.3249	64	0.8988	0.4384	2.0503
19	0.3256	0.9455	0.3443	65	0.9063	0.4226	2.1445
20	0.3420	0.9397	0.3640	66	0.9135	0.4067	2.2460
21	0.3584	0.9336	0.3839	67	0.9205	0.3907	2.3559
22	0.3746	0.9272	0.4040	68	0.9272	0.3746	2.4751
23	0.3907	0.9205	0.4245	69	0.9336	0.3584	2.6051
24	0.4067	0.9135	0.4452	70	0.9397	0.3420	2.7475
25	0.4226	0.9063	0.4663	71	0.9455	0.3256	2.9042
26	0.4384	0.8988	0.4877	72	0.9511	0.3090	3.0777
27	0.4540	0.8910	0.5095	73	0.9563	0.2924	3.2709
28	0.4695	0.8829	0.5317	74	0.9613	0.2756	3.4874
29	0.4848	0.8746	0.5543	75	0.9659	0.2588	3.7321
30	0.5000	0.8660	0.5774	76	0.9703	0.2419	4.0108
31	0.5150	0.8572	0.6009	77	0.9744	0.2205	4.3315
32	0.5299	0.8480	0.6249	78	0.9781	0.2079	4.7046
33	0.5446	0.8387	0.6494	79	0.9816	0.1908	5.1446
34	0.5592	0.8290	0.6745	80	0.9848	0.1736	5.6713
35	0.5736	0.8192	0.7002	81	0.9877	0.1564	6.3138
36	0.5878	0.8090	0.7265	82	0.9903	0.1392	7.1154
37	0.6018	0.7986	0.7536	83	0.9925	0.1219	8.1443
38	0.6157	0.7880	0.7813	84	0.9945	0.1045	9.5144
39	0.6293	0.7771	0.8098	85	0.9962	0.0872	11.4301
40	0.6428	0.7660	0.8391	86	0.9976	0.0698	14.3007
41	0.6561	0.7547	0.8693	87	0.9986	0.0523	19.0811
42	0.6691	0.7431	0.9004	88	0.9994	0.0349	28.6363
43	0.6820	0.7314	0.9325	89	0.9998	0.0175	57.2900

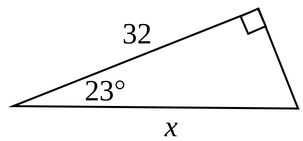
44	0.6947	0.7193	0.9657	90	1.0000	0.0000	------

Ex: Solve for the missing side in each of the following diagrams.

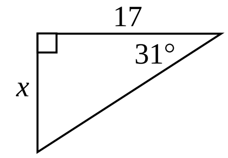
a.



b.

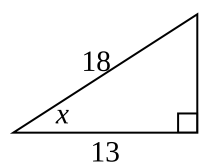


c.



	a.	b.	c.
1. Identify *acute* angle	55°	23°	31°
2. Name sides of interest	Hypotenuse = 20 Opposite = x	Adjacent = 32 Hypotenuse = x	Adjacent = 17 Opposite = x
3. Choose appropriate ratio	Use Opp/Hyp	Use Adj/Hyp	Use Opp/Adj
4. Set up proportion	$\frac{x}{20} = \frac{Opp}{Hyp}$	$\frac{32}{x} = \frac{Adj}{Hyp}$	$\frac{x}{17} = \frac{Opp}{Adj}$
5. Look up ratio in table	$\frac{x}{20} = 0.8192$	$\frac{32}{x} = 0.9205$	$\frac{x}{17} = 0.6009$
6. Solve	x = 20(0.8192) = 16.384	0.9205x = 32 x = 34.764	x = 17(0.6009) = 10.215

Ex: Find the measure of the angle marked *x* to the nearest degree.



Acute angle is x.

Adjacent = 13, hypotenuse = 18.

Use Adj/Hyp

$$\frac{13}{18} = \frac{Adj}{Hyp}$$

$$0.7222 = \frac{Adj}{Hyp}$$

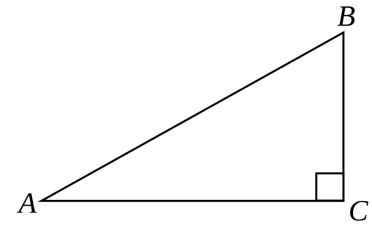
Look under Adj/Hyp in the table to find x is between 43° and 44°; closer to 44°

x ≈ 44°

Geometry Notes Trigonometry – 2: Trig Ratios

<u>Trigonometric Ratios</u>

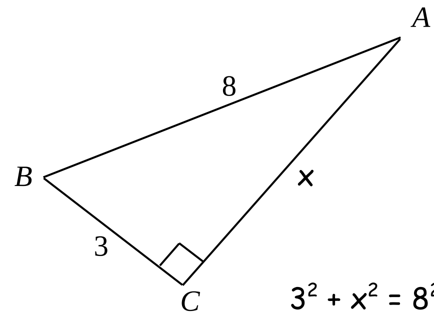
Definitions: For an *acute* angle in a *right triangle*,



$\sin\angle = \dfrac{\text{opp}}{\text{hyp}}$ $\qquad$ $\cos\angle = \dfrac{\text{adj}}{\text{hyp}}$ $\qquad$ $\tan\angle = \dfrac{\text{opp}}{\text{adj}}$

Memorize these!

Ex: Use the diagram at right to find the following trig ratios:



$3^2 + x^2 = 8^2$

$x^2 = 55$

$x = \sqrt{55}$

$\sin A = \dfrac{3}{8}$

$\sin B = \dfrac{\sqrt{55}}{8}$

$\sin C =$ C is NOT an acute angle. This is A2T.

$\cos A = \dfrac{\sqrt{55}}{8}$

$\cos B = \dfrac{3}{8}$

$\cos C =$ C is NOT an acute angle. This is A2T.

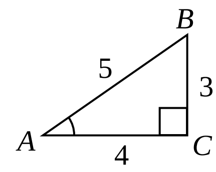
$\tan A = \dfrac{3}{\sqrt{55}}$

$\tan B = \dfrac{\sqrt{55}}{3}$

$\tan C =$ C is NOT an acute angle. This is A2T.

Note: By themselves, sin, cos and tan have *no mathematical meaning*. They *must include an angle*. For example,
"tan *A*" means "the tangent of angle *A*" and "tan60°" means "the tangent of a 60° angle." But "tan" just means "light brown."

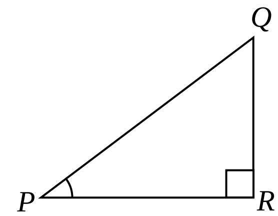
Ex: a. Use the diagram at right to find the following:



sin A = $\frac{3}{5}$ = 0.6 cos A = $\frac{4}{5}$ = 0.8

tan A = $\frac{3}{4}$ = 0.75

b. In ΔPQR, R is a right angle and $\angle P \cong \angle A$ above but we don't know any of the sides of the triangle. Can we find $\sin P$, $\cos P$ and $\tan P$?



Note that $\Delta ABC \sim \Delta PQR$ by AA.

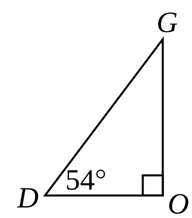
sin P = $\frac{QR}{QP} = \frac{BC}{BA}$ = 0.6 cos P = $\frac{RP}{QP} = \frac{CA}{BA}$ = 0.8

tan P = $\frac{RP}{QP} = \frac{CA}{BA}$ = 0.75

Important: The trig ratio values of an angle depend only on the *measure* of the angle, *not* on the size of the triangle used.

All angles of the same measure will have the same trig ratios. This is why Hipp's table is so useful.

Ex: In the triangle at right, find



sin D = 0.8090 cos D = 0.5878 tan D = 1.3764

sin G = 0.5878 cos G = 0.8090 tan G = 0.7265

Evaluating Trig Functions

Exact values of the trig ratios can be found for certain special angles: 0°, 30°, 45°, 60°, 90° (future lesson). For most purposes, decimal *approximations* are good enough. Your calculator can find approximations for the trig function ratios for any angle.

IMPORTANT: MAKE SURE YOUR CALCULATOR IS IN DEGREE MODE:

MODE ⬇⬇➡ (Should have **DEGREE** highlighted**) ENTER 2ND QUIT**

Ex: Use your calculator to evaluate the following to the nearest thousandth:

sin 67° = 0.9205 cos 48° = 0.6691 tan 78° = 4.7046

Geometry Notes Trigonometry – 3: Inverse Trig Functions

<u>Review</u>

Ex: Solve for x

a. $7 + 4 = x$ x = 11 x "by itself." Do given operation (add).

b. $x + \cancel{4} = 7$ x = 3 x not "by itself." Solve for x by doing OPPOSITE operation (subtract).

$$\begin{array}{r} x + \cancel{4} = 7 \\ -\cancel{4} \quad -4 \\ \hline x = 3 \end{array}$$

c. $3 + 2(11) = x$ x = 25 x "by itself." Do given operations in ORDER OF OPS (multiply first, then add).

d. $\cancel{3} + 2x = 11$ x = 4 x not "by itself." Do OPPOSITE operations in OPPOSITE ORDER of ops (subtract first, then divide).

$$\begin{array}{r} \cancel{3} + 2x = 11 \\ -\cancel{3} \qquad -3 \\ \hline \frac{\cancel{2}x}{\cancel{2}} = \frac{8}{2} \\ x = 4 \end{array}$$

Equations: If x by itself, do given operations in order of operations.
If x not by itself,
a. do OPPOSITE OPERATIONS in OPPOSITE ORDER.
b. do SAME THING on BOTH SIDES of equation

<u>Inverse Trig Functions</u>

Same idea with trig functions:

Ex: Solve for x:

a. $\sin 31° = x$ x by itself. Do $\sin(31°) = 0.5150$.

b. $x = \cos 42°$ x by itself. Do $\cos(42°) = 0.7431$.

c. $\sin x = 0.829$ x not by itself. Need to do the opposite of sine.

Definition: the "opposite" (inverse) of the sine function is called "inverse sine:"

Opposite of sin is $\sin^{-1}$ (also sometimes written arcsin)

Ex: Solve $\sin x = 0.829$

Do $\sin^{-1}$ on both sides: $\cancel{\sin^{-1}}(\cancel{\sin}\, x) = \sin^{-1}(0.829)$

$x = 55.996° \approx 56°$

Ex: Solve for x:

a. $\tan x = 1.35$ Use $\tan^{-1}$: $\tan^{-1}(\tan x) = \tan^{-1}(1.35)$

$x \approx 53.5°$

b. $\cos x = \frac{2}{3}$ Use $\cos^{-1}$: $\cos^{-1}(\cos x) = \cos^{-1}(2/3)$

$x \approx 48.2°$

c. $x = \sin 85°$ x by itself, use sin: $\sin(58°) \approx 0.848$

d. $\sin x = 1.25$ Use $\sin^{-1}$: $\sin^{-1}(\sin x) = \sin^{-1}(1.25)$

x = ERROR (sin x can never be greater than 1.) No solution.

Simple Trig Equations

Ex: Solve for x:

a. $\frac{\cos 72^{\circ}}{1} = \frac{x}{14}$ Cross multiply.

$x = 14\cos(72°) \approx 4.326$

b. $\frac{\sin 38^{\circ}}{1} = \frac{12}{x}$ Cross multiply.

$\frac{x\sin(38°)}{\sin(38°)} = \frac{12}{\sin(38°)}$

$x \approx 19.5$

c. $\cos x = \frac{3}{7}$ Do NOT cross multiply. Will not help get x by itself.

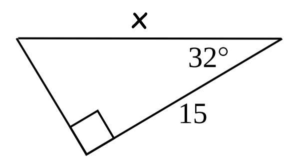
$\cos^{-1}(\cos x) = \cos^{-1}(3/7) \approx 64.6°$

Geometry Notes Trigonometry – 4: Solving Triangles

To find unknown part(s) of a right triangle:

1. *Draw* appropriate **RIGHT** triangle (if diagram not given.)
2. *Label* the **KNOWN** side(s) and angle(s); label the UNKNOWN (*x*).
3. *Write* appropriate trig ratio *equation*.
4. *Solve*.

Ex: Find the length of the hypotenuse in the triangle at right.



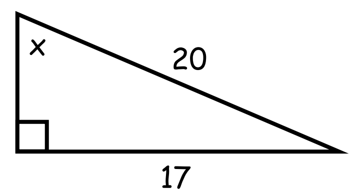
Acute angle: 32°

15 = adjacent, x = hypotenuse; use cos: $\cos(32^\circ) = \dfrac{15}{x}$

$$x\cos(32^\circ) = 15$$

$$X = \frac{15}{\cos(32^\circ)} \approx 17.7$$

Ex: A right triangle has hypotenuse 20 and its longer leg measures 17. Find the measure of the larger acute angle.



DRAW A DIAGRAM!
Angle is x

17 = opposite, 20 = hypotenuse; use sin: $\sin x = \dfrac{17}{20}$

$$x = \sin^{-1}\left(\frac{17}{20}\right) \approx 58.2^\circ$$

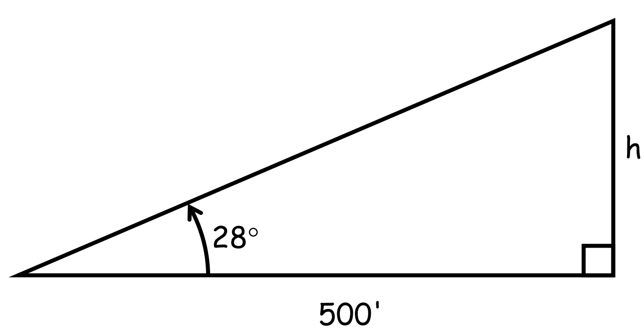
Ex: The angle of elevation to the top of a cliff from a point on level ground 500' from the base of the cliff is 28°. Find, to the nearest foot, the height of the cliff.

DRAW A DIAGRAM!
Angle is 28°
h = opposite, 500 = adjacent; use tan:

$$\tan(28^\circ) = \frac{h}{500}$$

$h = 500\tan(28^\circ) \approx 265.8$

Cliff is about 266 feet high.



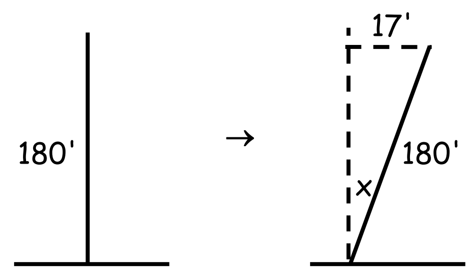
Note: "Angle of elevation" is always measured UP from the HORIZONTAL.

"Angle of depression" is always measured DOWN from the horizontal.

Geometry Notes Trigonometry – 5: Word Problems

Ex: When built, the Tower of Pisa stood 180 feet tall. Now the top is 17 feet away from the vertical. What angle does the tower make with the vertical?

DRAW A DIAGRAM!
Angle is x
17 = opposite, 180 = hypotenuse; use sin:



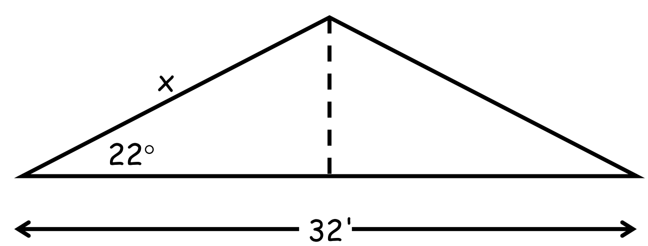
$$\sin x = \frac{17}{180}$$

$x = \sin^{-1}(17/180) \approx 5.4^\circ$

Tower leans about 5.4° from the vertical.

Ex: A peaked roof slopes up at an angle of 22°. If the entire roof (including eaves) is 32 feet wide, how long is each rafter?

DRAW A DIAGRAM!
Angle is 22°
x = hypotenuse, 16 = adjacent; use cos:



Note: This is NOT a right triangle. We need to fix that problem. Draw an altitude.

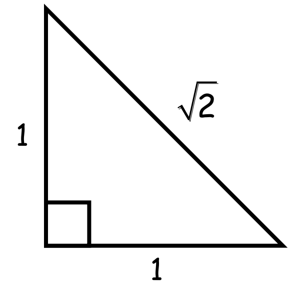
$$\cos(22^\circ) = \frac{16}{x}$$

$x\cos(22^\circ) = 16$

$$x = \frac{16}{\cos(22^\circ)} \approx 17.25 \text{ feet}$$

Geometry Class Work (Day 6): Special Angles and Cofunctions

Trig Ratios for Special Right Triangles



1. a. Draw an isosceles right triangle. Let the legs measure 1 unit.

   b. Find the measures of the acute angles.

   45°

   c. Find the length of the hypotenuse (*leave in radical form*).

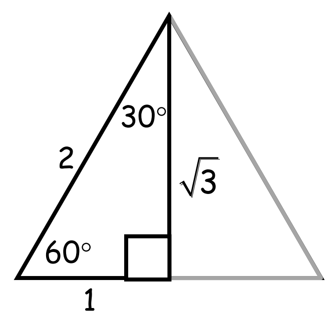
   $1^2 + 1^2 = x^2$

   $2 = x^2$

   $x = \sqrt{2}$

   d. Use your triangle (*not* your calculator!) to find
   the exact values (in fraction form) of the following:

   sin 45°= $\frac{1}{\sqrt{2}}$ cos 45° = $\frac{1}{\sqrt{2}}$ tan 45° = 1



2. a. Draw (preferably with a pencil) an equilateral triangle.
   Let each side measure 2. Draw an altitude. You now
   have two congruent triangles. Keep one; erase the other.

   b. Find the measures of the acute angles.

   60° and 30°

   b. Find the length of the two legs (leave the longer leg in radical form).

   Short leg = 1. Long leg: $1^2 + x^2 = 2^2$

   $x^2 = 3$

   $x = \sqrt{3}$

   c. Use your triangle (*not* your calculator!) to find
   the exact values (in fraction form) of the following:

   sin 30°= $\frac{1}{2}$ cos 30° = $\frac{\sqrt{3}}{2}$ tan 30° = $\frac{1}{\sqrt{3}}$

   sin 60°= $\frac{\sqrt{3}}{2}$ cos 60° = $\frac{1}{2}$ tan 60° = $\sqrt{3}$

You now have a choice. Either

1. *Memorize* the nine special trig function values you have found above *OR*

2. *Memorize* the two special triangles above so you can use them to quickly find the special trig function values.

(*Continue on other side.*)

Cofunction Relations

Sine and cosine are a pair of *cofunctions*. (So are tangent and cotangent, and secant and cosecant: A2T)

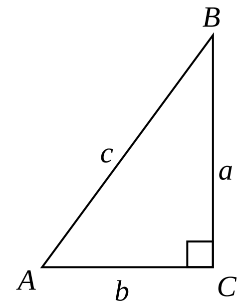
1. In the diagram at right, what is the relationship between angles A and B (a single vocabulary word)?

   Complementary

2. In the diagram at right, find the values of

   $\sin A = \frac{a}{c}$ $\cos A = \frac{a}{c}$

   $\sin B = \frac{b}{c}$ $\cos B = \frac{b}{c}$



3. If x is any acute angle and y is its complement, then

   $\sin x =$ cosy $\cos x =$ sinx

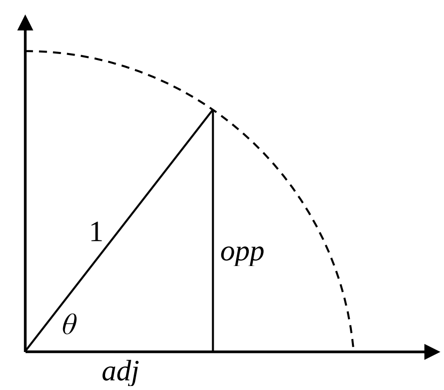
4. Conversely, if x and y are acute angles and $\sin x = \cos y$, then

   $x + y =$
   90°

Trig Ratios for Quadrantal Angles

Angles of 0° and 90° are called *quadrantal angles* (so are 180°, 270° and 360° but you can worry about those next year). Your calculator can find trig ratio values for those angles but this will help you understand them.

Imagine a right triangle whose hypotenuse is always 1 and with an acute angle that can vary between 0° and 90°.



1. Let θ get smaller and smaller. Consider what the "triangle" would look like when $\theta = 0°$

   a. What is the length of the "opposite side"? 0

   b. What is the length of the "adjacent side"? 1

   c. Using your answers above, what are logical values for the following?

   0
   0

   $\sin 0° =$ 0 $\cos 0° =$ 1 $\tan 0° =$ 0

2. Now let θ get larger and larger. Consider what the "triangle" would look like when $\theta = 90°$

   a. What is the length of the "opposite side"? 1

   b. What is the length of the "adjacent side"? 0

   c. Using your answers above, what are logical values for the following?

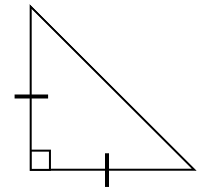
   $\sin 90° =$ 1 $\cos 90° =$ 0 $\tan 90° =$ undefined 1/0

Geometry Notes Trigonometry – 6: Special Angles and Cofunctions

Special Acute (Tri)Angles

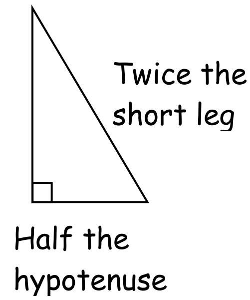
45°–45°–90°

Fact: Legs are congruent.

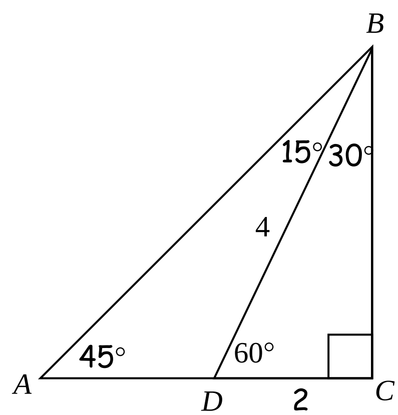


30°–60°–90°

Fact: Short leg = half the hypotenuse
(hypotenuse = twice the short leg: h = 2ℓ)



Ex: In the diagram, $DB = 4$, $m\angle BDC = 60°$ and $\overline{AC} \cong \overline{BC}$.
Find the values of



a. $m\angle DBC$ = 30°

b. $m\angle A$ = 45°

c. $m\angle DBA$ = 45° - 30° = 15°

d. DC = $\frac{1}{2}(4) = 2$

e. CB $2^2 + (CB)^2 = 4^2$ so $CB = \sqrt{12}$

f. AC = $BC = \sqrt{12}$

g. AB $(\sqrt{12})^2 + (\sqrt{12})^2 = (AB)^2$ so $AB = \sqrt{24}$

h. AD = $\sqrt{12} - 2$

Cofunctions

The *co* in cosine is short for *complementary*. So cosine means "complementary sine" or, more precisely, "the sine of the complement."

Ex: Find a value of x if $\sin x = \cos(3x - 10)$

If sinA = cosB, then A + B = 90. Know this.

$$x + 3x - 10 = 90$$
$$4x = 100$$
$$x = 25$$